STACKS

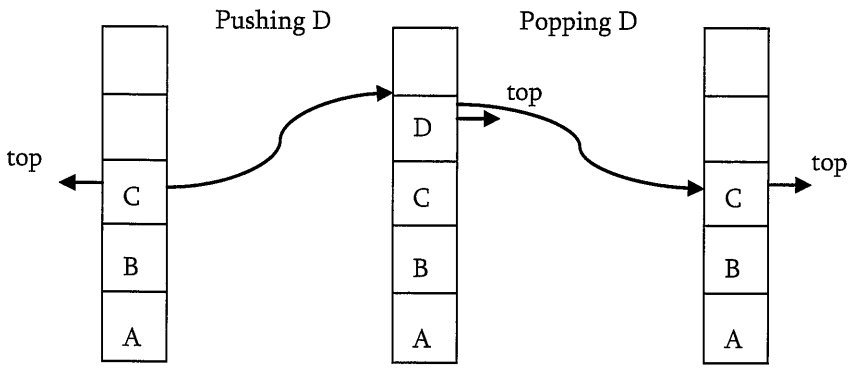
Chapter-4

4.1 What is a Stack?

A stack is a simple data structure used for storing data (similar to Linked Lists). In stack, the order in which the data arrives is important. The pile of plates of a cafeteria is a good example of stack. The plates are added to the stack as they are cleaned. They are placed on the top. When a plate is required it is taken from the top of the stack. The first plate placed on the stack is the last one to be used.

Definition: A *stack* is an ordered list in which insertion and deletion are done at one end, where the end is called as *top*. The last element inserted is the first one to be deleted. Hence, it is called Last in First out (LIFO) or First in Last out (FILO) list.

Special names are given to the two changes that can be made to a stack. When an element is inserted in a stack, the concept is called as *push*, and when an element is removed from the stack, the concept is called as *pop*. Trying to pop out an empty stack is called as *underflow* and trying to push an element in a full stack is called as *overflow*. Generally, we treat them as exceptions. As an example, consider the snapshots of the stack.



4.2 How Stacks are used?

Consider a working day in the office. Let us assume a developer is working on a long-term project. The manager then gives the developer a new task, which is more important. The developer places the long-term project aside and begins work on the new task. The phone then rings, this is the highest priority, as it must be answered immediately. The developer pushes the present task into the pending tray and answers the phone. When the call is complete the task abandoned top answer the phone is retrieved from the pending tray and work progresses. If another call comes in, it may have to be handled in the same manner, but eventually the new task will be finished, and the developer can draw the long-term project from the pending tray and continue with that.

4.3 Stack ADT

The following operations make a stack an ADT. For simplicity assume the data is of integer type.

Main stack operations

- Push (int data): Inserts *data* onto stack.

- int Pop(): Removes and returns the last inserted element from the stack.

Auxiliary stack operations

- int Top(): Returns the last inserted element without removing it.
- int Size(): Returns the number of elements stored in stack.
- int IsEmptyStack(): Indicates whether any elements are stored in stack or not.
- int IsFullStack(): Indicates whether the stack is full or not.

Exceptions

Attempting the execution of an operation may sometimes cause an error condition, called an exception. Exceptions are said to be "thrown" by an operation that cannot be executed. In the Stack ADT, operations pop and top cannot be performed if the stack is empty. Attempting the execution of pop (top) on an empty stack throws an exception. Trying to push an element in a full stack throws an exception.

4.4 Applications

Following are the some of the applications in which stacks plays an important role.

Direct applications

- Balancing of symbols
- Infix-to-postfix conversion
- Evaluation of postfix expression
- Implementing function calls (including recursion)
- Finding of spans (finding spans in stock markets, refer *Problems* section)
- Page-visited history in a Web browser [Back Buttons]
- Undo sequence in a text editor
- Matching Tags in HTML and XML

Indirect applications

- Auxiliary data structure for other algorithms (Example: Tree traversal algorithms)
- Component of other data structures (Example: Simulating queues, refer *Queues* chapter)

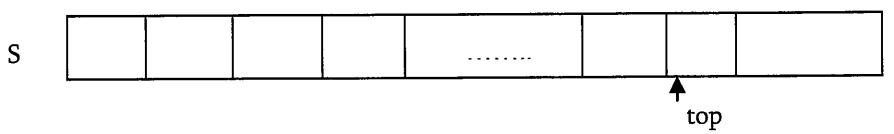
4.5 Implementation

There are many ways of implementing stack ADT and below are the commonly used methods.

- Simple array based implementation
- Dynamic array based implementation
- Linked lists implementation

Simple Array Implementation

This implementation of stack ADT uses an array. In the array, we add elements from left to right and use a variable to keep track of the index of the top element.



The array storing the stack elements may become full. A push operation will then throw a *full stack exception*. Similarly, if we try deleting an element from empty stack then it will throw *stack empty exception*.

```
struct ArrayStack {
        int top;
        int capacity;
        int *array;
};
struct ArrayStack *CreateStack() {
        struct ArrayStack *S = malloc(sizeof(struct ArrayStack));
        if(!S) return NULL;
        S→capacity = 1;
        S→top = -1;
        S→array= malloc(S→capacity * sizeof(int));
        if(!S→array) return NULL;
        return S;
}
int IsEmptyStack(struct ArrayStack *S) {
        return (S→top == -1);      // if the condition is true then 1 is returned else 0 is returned
}
int IsFullStack(struct ArrayStack *S){
        //if the condition is true then 1 is returned else 0 is returned
        return (S→top == S→capacity - 1);
}
void Push(struct ArrayStack *S, int data){
        /* S→top == capacity -1 indicates that the stack is full*/
        if(IsFullStack(S)) printf( "Stack Overflow");
        else      /*Increasing the 'top' by 1 and storing the value at 'top' position*/
                S→ array[++S→top]= data;
}
int Pop(struct ArrayStack *S){
        if(IsEmptyStack(S)){      /* S→top == - 1 indicates empty stack*/
                printf("Stack is Empty");
                return 0;
        }
        else /* Removing element from 'top' of the array and reducing 'top' by 1*/
                return (S→ array[S→top--]);
}
void DeleteStack(struct DynArrayStack *S){
        if(S) {   if(S→array) free(S→array);
                free(S);
        }
}
```

Performance & Limitations

Performance

Let n be the number of elements in the stack. The complexities of stack operations with this representation can be given as:

Space Complexity (for n push operations)	$O(n)$

Time Complexity of Push()	O(1)
Time Complexity of Pop()	O(1)
Time Complexity of Size()	O(1)
Time Complexity of IsEmptyStack()	O(1)
Time Complexity of IsFullStack()	O(1)
Time Complexity of DeleteStack()	O(1)

Limitations

The maximum size of the stack must be defined in prior and cannot be changed. Trying to push a new element into a full stack causes an implementation-specific exception.

Dynamic Array Implementation

First, let's consider how we implemented a simple array based stack. We took one index variable *top* which points to the index of the most recently inserted element in the stack. To insert (or push) an element, we increment *top* index and then place the new element at that index. Similarly, to delete (or pop) an element we take the element at *top* index and then decrement the *top* index. We represent empty queue with *top* value equal to -1. The issue still need to be resolved is that what we do when all the slots in fixed size array stack are occupied?

First try: What if we increment the size of the array by 1 every time the stack is full?

- Push(): increase size of S[] by 1
- Pop(): decrease size of S[] by 1

Problems with this approach?

This way of incrementing the array size is too expensive. Let us a see the reason for this. For example, at $n = 1$, to push an element create a new array of size 2 and copy all the old array elements to new array and at the end add the new element. At $n = 2$, to push an element create a new array of size 3 and copy all the old array elements to new array and at the end add the new element.

Similarly, at $n = n - 1$, if we want to push an element create a new array of size n and copy all the old array elements to new array and at the end add the new element. After n push operations the total time $T(n)$ (number of copy operations) is proportional to $1 + 2 + \ldots + n \approx O(n^2)$.

Alternative Approach: Repeated Doubling

Let us improve the complexity by using array *doubling* technique. If the array is full, create a new array of twice the size, and copy items. With this approach, pushing n items takes time proportional to n (not n^2).

For simplicity, let us assume that initially we started with $n = 1$ and moved till $n = 32$. That means, we do the doubling at $1, 2, 4, 8, 16$. The other way of analyzing the same is, at $n = 1$, if we want to add (push) an element then double the current size of array and copy all the elements of old array to new array.

At, $n = 1$, we do 1 copy operation, at $n = 2$, we do 2 copy operations, and n = 4, we do 4 copy operations and so on. By the time we reach $n = 32$, the total number of copy operations is $1 + 2 + 4 + 8 + 16 = 31$ which is approximately equal to $2n$ value (32). If we observe carefully, we are doing the doubling operation $logn$ times.

Now, let us generalize the discussion. For n push operations we double the array size $logn$ times. That means, we will have $logn$ terms in below expression. The total time $T(n)$ of a series of n push operations is proportional to

$$1 + 2 + 4 + 8 \ldots + \frac{n}{4} + \frac{n}{2} + n = n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} \ldots + 4 + 2 + 1$$
$$= n\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots + \frac{4}{n} + \frac{2}{n} + \frac{1}{n}\right)$$
$$= n(2) \approx 2n = O(n)$$

$T(n)$ is $O(n)$ and the amortized time of a push operation is $O(1)$.

```
struct DynArrayStack {
        int top;
        int capacity;
        int *array;
};
struct DynArrayStack *CreateStack(){
        struct DynArrayStack *S = malloc(sizeof(struct DynArrayStack));
        if(!S) return NULL;
        S→capacity = 1;
        S→top = -1;
        S→array = malloc(S→capacity * sizeof(int));          // allocate an array of size 1 initially
        if(!S→array) return NULL;
        return S;
}
int IsFullStack(struct DynArrayStack *S){
        return (S→top == S→capacity-1);
}
void DoubleStack(struct DynArrayStack *S){
        S→capacity *= 2;
        S→array = realloc(S→array, S→capacity);
}
void Push(struct DynArrayStack *S, int x){
        // No overflow in this implementation
        if(IsFullStack(S))
                DoubleStack(S);
        S→array[++S→top] = x;
}
int IsEmptyStack(struct DynArrayStack *S){
        return S→top == -1;
}
int Top(struct DynArrayStack *S){
        if(IsEmptyStack(S))
                return INT_MIN;
        return S→array[S→top];
}
int Pop(struct DynArrayStack *S){
        if(IsEmptyStack(S))
                return INT_MIN;
        return S→array[S→top--];
}
void DeleteStack(struct DynArrayStack *S){
        if(S) {
                if(S→array) free(S→array);
                free(S);
        }
}
```

Performance

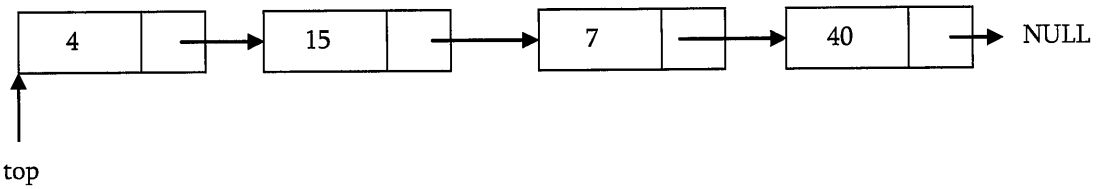
Let n be the number of elements in the stack. The complexities for operations with this representation can be given as:

Space Complexity (for n push operations)	$O(n)$
Time Complexity of CreateStack()	O(1)
Time Complexity of Push()	O(1) (Average)
Time Complexity of Pop()	O(1)
Time Complexity of Top()	O(1)
Time Complexity of IsEmptyStack()	O(1))
Time Complexity of IsFullStack()	O(1)
Time Complexity of DeleteStack()	O(1)

Note: Too many doublings may cause memory overflow exception.

Linked List Implementation

The other way of implementing stacks is by using Linked lists. Push operation is implemented by inserting element at the beginning of the list. Pop operation is implemented by deleting the node from the beginning (the header/top node).




```
struct ListNode{
        int data;
        struct ListNode *next;
};
struct Stack *CreateStack(){
        return NULL;
}
void Push(struct Stack **top, int data){
         struct Stack *temp;
         temp = malloc(sizeof(struct Stack));
        if(!temp) return NULL;
        temp→data = data;
        temp→next = *top;
        *top = temp;
}
int IsEmptyStack(struct Stack *top){
        return top == NULL;
}
int Pop(struct Stack **top){
        int data;
        struct Stack *temp;
        if(IsEmptyStack(top))
                return INT_MIN;
        temp = *top;
        *top = *top→next;
        data = temp→data;
        free(temp);
        return data;
```

```
}
int Top(struct Stack * top){
        if(IsEmptyStack(top))  return INT_MIN;
        return top→next→data;
}
void DeleteStack(struct Stack **top){
        struct Stack *temp, *p;
        p = *top;
        while( p→next) {
                temp = p→next;
                p→next = temp→next;
                free(temp);
        }
        free(p);
}
```

Performance

Let n be the number of elements in the stack. Let n be the number of elements in the stack. The complexities for operations with this representation can be given as:

Space Complexity (for n push operations)	$O(n)$
Time Complexity of CreateStack()	$O(1)$
Time Complexity of Push()	$O(1)$ (Average)
Time Complexity of Pop()	$O(1)$
Time Complexity of Top()	$O(1)$
Time Complexity of IsEmptyStack()	$O(1)$
Time Complexity of DeleteStack()	$O(n)$

4.6 Comparison of Implementations

Comparing Incremental Strategy and Doubling Strategy

We compare the incremental strategy and doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n push operations. We start with an empty stack represented by an array of size 1. We call *amortized* time of a push operation is the average time taken by a push over the series of operations, i.e., $T(n)/n$.

Incremental Strategy: The amortized time (average time per operation) of a push operation is $O(n)$ [$O(n^2)/n$].

Doubling Strategy: In this method, the amortized time of a push operation is $O(1)$ [$O(n) / n$].

Note: For reasoning, refer implementation section.

Comparing Array Implementation and Linked List Implementation

Array Implementation

- Operations take constant time.
- Expensive doubling operation every once in a while.
- Any sequence of n operations (starting from empty stack) -- "amortized" bound takes time proportional to n.

Linked list Implementation

- Grows and shrinks gracefully.
- Every operation takes constant time $O(1)$.

- Every operation uses extra space and time to deal with references.

4.7 Problems on Stacks

Problem-1 Discuss how stacks can be used for checking balancing of symbols?

Solution: Stacks can be used to check whether the given expression has balanced symbols or not. This algorithm is very much useful in compilers. Each time parser reads one character at a time. If the character is an opening delimiter like (, {, or [- then it is written to the stack. When a closing delimiter is encountered like),}, or]- is encountered the stack is popped. The opening and closing delimiters are then compared. If they match, the parsing of the string continues. If they do not match, the parser indicates that there is an error on the line. A linear-time $O(n)$ algorithm based on stack can be given as:

Algorithm

a) Create a stack.
b) while (end of input is not reached) {
    1) If the character read is not a symbol to be balanced, ignore it.
    2) If the character is an opening symbol like (, [, {, push it onto the stack
    3) If it is a closing symbol like),],}, then if the stack is empty report an error. Otherwise pop the stack.
    4) If the symbol popped is not the corresponding opening symbol, report an error.
   }
c) At end of input, if the stack is not empty report an error

Examples:

Example	Valid?	Description
(A+B)+(C-D)	Yes	The expression is having balanced symbol
((A+B)+(C-D)	No	One closing brace is missing
((A+B)+[C-D])	Yes	Opening and immediate closing braces correspond
((A+B)+[C-D]}	No	The last closing brace does not correspond with the first opening parenthesis

For tracing the algorithm let us assume that the input is: () (() [()])

Input Symbol, A[i]	Operation	Stack	Output
(	Push (	(	
)	Pop (Test if (and A[i] match? YES		
(	Push (	(	
(	Push (	((	
)	Pop (Test if (and A[i] match? YES	(	
[	Push [	([	
(	Push (	([(	
)	Pop (Test if(and A[i] match? YES	([	
]	Pop [Test if [and A[i] match? YES	(	

)	Pop (Test if(and A[i] match? YES		
	Test if stack is Empty? YES		TRUE

Time Complexity: $O(n)$. Since, we are scanning the input only once. Space Complexity: $O(n)$ [for stack].

Problem-2 Discuss infix to postfix conversion algorithm using stack?

Solution: Before discussing the algorithm, first let us see the definitions of infix, prefix and postfix expressions.

Infix: An infix expression is a single letter, or an operator, proceeded by one infix string and followed by another Infix string.

A
A+B
(A+B)+ (C-D)

Prefix: A prefix expression is a single letter, or an operator, followed by two prefix strings. Every prefix string longer than a single variable contains an operator, first operand and second operand.

A
+AB
++AB-CD

Postfix: A postfix expression (also called Reverse Polish Notation) is a single letter or an operator, preceded by two postfix strings. Every postfix string longer than a single variable contains first and second operands followed by an operator.

A
AB+
AB+CD-+

Prefix and postfix notions are methods of writing mathematical expressions without parenthesis. Time to evaluate a postfix and prefix expression is $O(n)$, were n is the number of elements in the array.

Infix	Prefix	Postfix
A+B	+AB	AB+
A+B-C	-+ABC	AB+C-
(A+B)*C-D	-*+ABCD	AB+C*D-

Now, let us concentrate on the algorithm. In infix expressions, the operator precedence is implicit unless we use parentheses. Therefore, for the infix to postfix conversion algorithm we have to define the operator precedence (or priority) inside the algorithm. The table shows the precedence and their associatively (order of evaluation) among operators.

Token	Operator	Precedence	Associatively
( ) [ ] → .	function call array element struct or union member	17	left-to-right
-- ++	increment, decrement	16	left-to-right
-- ++ ! ~ - + & * sizeof	decrement, increment logical not one's complement unary minus or plus address or indirection size (in bytes)	15	right-to-left
(type)	type cast	14	right-to-left
* / %	multiplicative	13	Left-to-right

+ -	binary add or subtract	12	left-to-right
<< >>	shift	11	left-to-right
> >= < <=	relational	10	left-to-right
== !=	equality	9	left-to-right
&	bitwise and	8	left-to-right
^	bitwise exclusive or	7	left-to-right
|	bitwise or	6	left-to-right
&&	logical and	5	left-to-right
||	logical or	4	left-to-right
?:	conditional	3	right-to-left
= += -= /= *= %= <<= >>= &= ^=	assignment	2	right-to-left
,	Comma	1	left-to-right

Important Properties

- Let us consider the infix expression 2 + 3 * 4 and its postfix equivalent 2 3 4 * +. Notice that between infix and postfix the order of the numbers (or operands) is unchanged. It is 2 3 4 in both cases. But the order of the operators * and + is affected in the two expressions.
- Only one stack is enough to convert an infix expression to postfix expression. The stack that we use in the algorithm will be used to change the order of operators from infix to postfix. The stack we use will only contain operators and the open parentheses symbol '('. Postfix expressions do not contain parentheses. We shall not output the parentheses in the postfix output.

Algorithm

a) Create a stack
b) for each character t in the input stream{
    if(t is an operand)
        output t
    else if(t is a right parenthesis){
        Pop and output tokens until a left parenthesis is popped (but not output)
    }
    else // t is an operator or left parenthesis{
        pop and output tokens until one of lower priority than t is encountered or a left parenthesis is encountered or the stack is empty
        Push t
    }
}
c) pop and output tokens until the stack is empty

For better understanding let us trace out some example: A * B- (C + D) + E

Input Character	Operation on Stack	Stack	Postfix Expression
A		Empty	A
*	Push	*	A
B		*	AB
-	Check and Push	-	AB*
(	Push	-(	AB*
C		-(	AB*C
+	Check and Push	-(+	AB*C

D			AB*CD
)	Pop and append to postfix till '('	-	AB*CD+
+	Check and Push	+	AB*CD+-
E		+	AB*CD+-E
End of input	Pop till empty		AB*CD+-E+

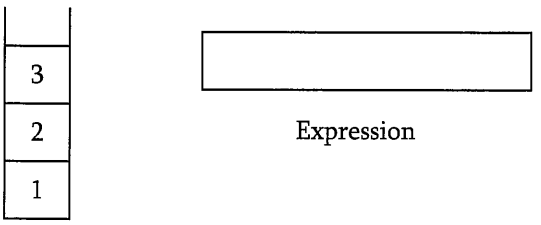
Problem-3 Discuss postfix evaluation using stacks?

Solution:

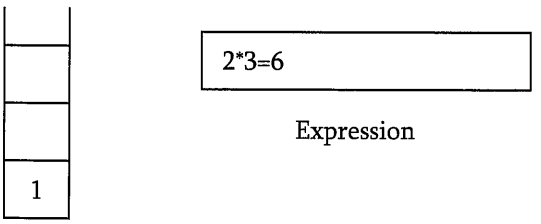
Algorithm

1 Scan the Postfix string from left to right.
2 Initialize an empty stack.
3 Repeat the below steps 4 and 5 till all the characters are scanned.
4 If the scanned character is an operand, push it onto the stack.
5 If the scanned character is an operator, and if the operator is unary operator then pop an element from the stack. If the operator is binary operator then pop two elements from the stack. After popping the elements, apply the operator to those popped elements. Let the result of this operation be retVal onto the stack.
6 After all characters are scanned, we will have only one element in the stack.
7 Return top of the stack as result.

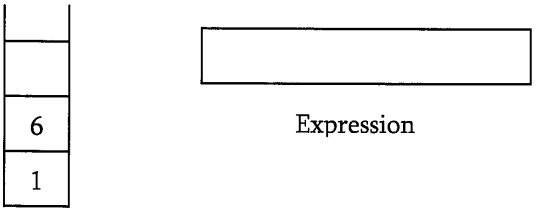
Example: Let us see how the above algorithm works using an example. Assume that the postfix string is 123*+5-. Initially the stack is empty. Now, the first three characters scanned are 1, 2 and 3, which are operands. They will be pushed into the stack in that order.



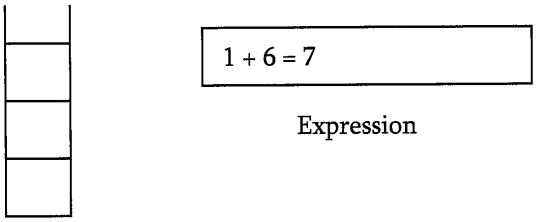
Next character scanned is "*", which is an operator. Thus, we pop the top two elements from the stack and perform the "*" operation with the two operands. The second operand will be the first element that is popped.



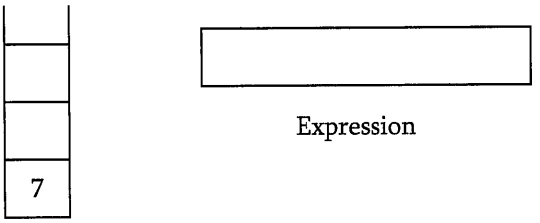
The value of the expression (2*3) that has been evaluated (6) is pushed into the stack.



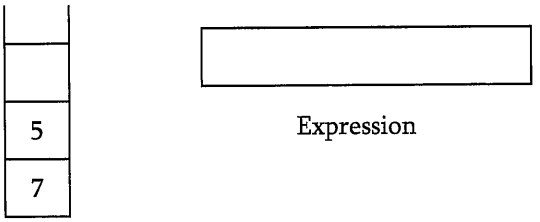
Next character scanned is "+", which is an operator. Thus, we pop the top two elements from the stack and perform the "+" operation with the two operands. The second operand will be the first element that is popped.



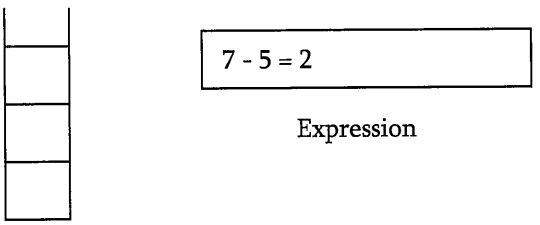
The value of the expression (1+6) that has been evaluated (7) is pushed into the stack.



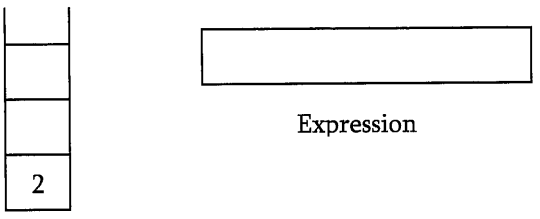
Next character scanned is "5", which is added to the stack.



Next character scanned is "-", which is an operator. Thus, we pop the top two elements from the stack and perform the "-" operation with the two operands. The second operand will be the first element that is popped.



The value of the expression(7-5) that has been evaluated(23) is pushed into the stack.



Now, since all the characters are scanned, the remaining element in the stack (there will be only one element in the stack) will be returned. End result:

- Postfix String : 123*+5-
- Result : 2

Problem-4 Can we evaluate the infix expression with stacks in one pass?

Solution: Using 2 stacks we can evaluate an infix expression in 1 pass without converting to postfix.

Algorithm

1) Create an empty operator stack
2) Create an empty operand stack
3) For each token in the input string
    a. Get the next token in the infix string
    b. If next token is an operand, place it on the operand stack
    c. If next token is an operator
        i. Evaluate the operator (next op)
4) While operator stack is not empty, pop operator and operands (left and right), evaluate left operator right and push result onto operand stack
5) Pop result from operator stack

Problem-5 How to design a stack such that GetMinimum( ) should be O(1)?

Solution: Take an auxiliary stack which maintains the minimum of all values in the stack. Also, assume that, each element of the stack is less than its below elements. For simplicity let us call the auxiliary stack as *min stack*.

When we *pop* the main stack, *pop* the min stack too. When we push the main stack, push either the new element or the current minimum, whichever is lower. At any point, if we want to get the minimum then we just need to return the top element from the min stack. Let us take some example and trace out. Initially let us assume that we have pushed 2, 6, 4, 1 and 5. Based on above algorithm the *min stack* will look like:

Main stack	Min stack
5 → top	1 → top
1	1
4	2
6	2
2	2

After popping twice we get:

Main stack	Min stack
4 → top	2 → top
6	2
2	2

Based on the above discussion, now let us code the push, pop and GetMinimum() operations.

```
struct AdvancedStack{
        struct Stack elementStack;
        struct Stack minStack;
};
void Push(struct AdvancedStack *S, int data ){
        Push (S→elementStack, data);
        if(IsEmptyStack(S→minStack) || Top(S→minStack) >= data)
                Push (S→minStack, data);
        else    Push (S→minStack, Top(S→minStack));
}
int Pop(struct AdvancedStack *S ){
```

```
        int temp;
        if(IsEmptyStack(S→elementStack))
                return -1;
        temp = Pop (S→elementStack);
        Pop (S→minStack);
        return temp;
}
int GetMinimum(struct AdvancedStack *S){
        return Top(S→minStack);
}
struct AdvancedStack *CreateAdvancedStack(){
        struct AdvancedStack *S = (struct AdvancedStack *)malloc(sizeof(struct AdvancedStack));
        if(!S)
                return NULL;
        S→elementStack = CreateStack();
        S→minStack = CreateStack();
        return S;
}
```

Time complexity: O(1). Space complexity: O(n) [for Min stack]. This algorithm has much better space usage if we rarely get a "new minimum or equal".

Problem-6 For the Problem-5 is it possible to improve the space complexity?

Solution: Yes. The main problem of previous approach is, for each push operation we are pushing the element on to *min stack* also (either the new element or existing minimum element). That means, we are pushing the duplicate minimum elements on to the stack.

Now, let us change the above algorithm to improve the space complexity. We still have the min stack, but we only pop from it when the value we pop from the main stack is equal to the one on the min stack. We only *push* to the min stack when the value being pushed onto the main stack is less than *or equa* to the current min value. In this modified algorithm also, if we want to get the minimum then we just need to return the top element from the min stack. For example, taking the original version and pushing 1 again, we'd get:

Main stack	Min stack
1 → top	
5	
1	
4	1 → top
6	1
2	2

Popping from the above pops from both stacks because 1 == 1, leaving:

Main stack	Min stack
5 → top	
1	
4	
6	1 → top
2	2

Popping again *only* pops from the main stack, because 5 > 1:

Main stack	Min stack
1 → top	
4	
6	1 → top

2	2

Popping again pops both stacks because 1 == 1:

Main stack	Min stack
4 → top	
6	
2	2 → top

Note: The difference is only in push & pop operations.

```
struct AdvancedStack {
        struct Stack elementStack;
        struct Stack minStack;
};
void Push(struct AdvancedStack *S, int data){
        Push (S→elementStack, data);
        if(IsEmptyStack(S→minStack) || Top(S→minStack) >= data)
                Push (S→minStack, data);
}
int Pop(struct AdvancedStack *S ){
        int temp;
        if(IsEmptyStack(S→elementStack)) return -1;
        temp = Top (S→elementStack);
        if(Top(S→ minStack) == Pop(S→elementStack))
                Pop (S→ minStack);
        return temp;
}
int GetMinimum(struct AdvancedStack *S){
        return Top(S→minStack);
}
Struct AdvancedStack * AdvancedStack(){
        struct AdvancedStack *S = (struct AdvancedStack) malloc (sizeof (struct AdvancedStack));
        if(!S) return NULL;
        S→elementStack = CreateStack();
        S→minStack = CreateStack();
        return S;
}
```

Time complexity: O(1). Space complexity: O(n) [for Min stack]. But this algorithm has much better space usage if we rarely get a "new minimum or equal".

Problem-7 For a given array with n symbols how many stack permutations are possible?

Solution: The number of stack permutations with n symbols is represented by *Catalan number* and we will discuss this in *Dynamic Programming* chapter.

Problem-8 Given an array of characters formed with a's and b's. The string is marked with special character X which represents the middle of the list (for example: ababa...ababXbabab.....baaa). Check whether the string is palindrome or not?

Solution: This is one of the simplest algorithms. What we do is, start two indexes one at the beginning of the string and other at the ending of the string. Each time compare whether the values at both the indexes are same or not. If the values are not same then we say that the given string is a palindrome. If the values are same then increment the left index and decrement the right index. Continue this process until both the indexes meet at the middle (at X) or if the string is not palindrome.

```
int IsPalindrome(char *A){
        int i=0, j = strlen(A)-1;
        while(i < j && A[i] == A[j]) {
                i++;
                j--;
        }
        if(i < j ) {
                printf("Not a Palindrome");
                return 0;
        }
        else {  printf("Palindrome");
                return 1;
        }
}
```

Time Complexity: $O(n)$. Space Complexity: $O(1)$.

Problem-9 For the Problem-8, if the input is in singly linked list then how do we check whether the list elements form a palindrome or not? (That means, moving backward is not possible).

Solution: Refer *Linked Lists* chapter.

Problem-10 Can we solve Problem-8 using stacks?

Solution: Yes.

Algorithm

- Traverse the list till we encounter X as input element.
- During the traversal push all the elements (until X) on to the stack.
- For the second half of the list, compare each elements content with top of the stack. If they are same then pop the stack and go to the next element in the input list.
- If they are not same then the given string is not a palindrome.
- Continue this process until the stack is empty or the string is not a palindrome.

```
int IsPalindrome(char *A){
        int i=0;
        struct Stack S= CreateStack();
        while(A[i] != 'X') {
                Push(S, A[i]);
                i++;
        }
        i++;
        while(A[i]) {
                if(IsEmptyStack(S) ||A[i] != Pop(S)) {
                        printf("Not a Palindrome");
                        return 0;
                }
                i++;
        }
        return IsEmptyStack(S);
}
```

Time Complexity: $O(n)$. Space Complexity: $O(n/2) \approx O(n)$.

Problem-11 Given a stack, how to reverse the elements of stack by using only stack operations (push & pop)?

Solution: Algorithm

- First pop all the elements of the stack till it becomes empty.
- For each upward step in recursion, insert the element at the bottom of stack.

```
void ReverseStack(struct Stack *S){
        int data;
        if(IsEmptyStack(S)) return;
        data = Pop(S);
        ReverseStack(S);
        InsertAtBottom(S, data);
}
void InsertAtBottom(struct Stack *S, int data){
        int temp;
        if(IsEmptyStack(S)) {
                Push(S, data);
                return;
        }
        temp = Pop(S);
        InsertAtBottom(S, data);
        Push(S, temp);
}
```

Time Complexity: $O(n^2)$. Space Complexity: $O(n)$, for recursive stack.

Problem-12 Show how to implement one queue efficiently using two stacks. Analyze the running time of the queue operations.

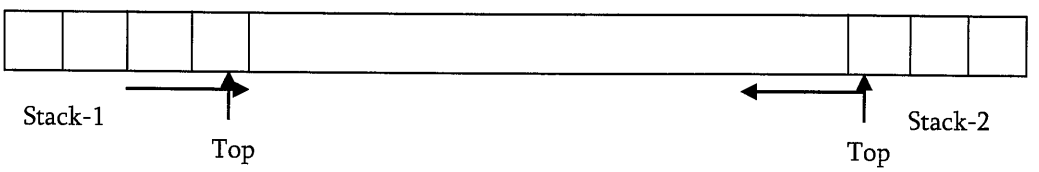
Solution: Refer *Queues* chapter.

Problem-13 Show how to implement one stack efficiently using two queues. Analyze the running time of the stack operations.

Solution: Refer *Queues* chapter.

Problem-14 How do we implement 2 stacks using only one array? Our stack routines should not indicate an exception unless every slot in the array is used?

Solution:



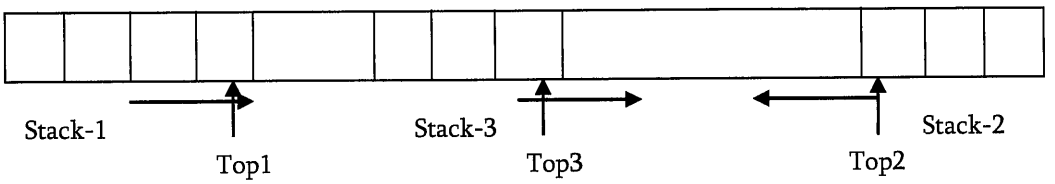
Algorithm:

- Start two indexes one at the left end and other at the right end.
- The left index simulates the first stack and the right index simulates the second stack.
- If we want to push an element into the first stack then put the element at left index.
- Similarly, if we want to push an element into the second stack then put the element at right index.
- First stack gets grows towards right, second stack grows towards left.

Time Complexity of push and pop for both stacks is O(1). Space Complexity is O(1).

Problem-15 3 stacks in one array: How to implement 3 stacks in one array?

Solution: For this problem, there could be other way of solving it. Below is one such possibility and it works as long as there is an empty space in the array.



To implement 3 stacks we keep the following information.

- The index of the first stack (Top1): this indicates the size of the first stack.
- The index of the second stack (Top2): this indicates the size of the second stack.
- Starting index of the third stack (base address of third stack).
- Top index of the third stack.

Now, let us define the push and pop operations for this implementation.

Pushing:

- For pushing on to the first stack, we need to see if adding a new element causes it to bump into the third stack. If so, try to shift the third stack upwards. Insert the new element at (start1 + Top1).
- For pushing to the second stack, we need to see if adding a new element causes it to bump into the third stack. If so, try to shift the third stack downward. Insert the new element at (start2 – Top2).
- When pushing to the third stack, see if it bumps the second stack. If so, try to shift the third stack downward and try pushing again. Insert the new element at (start3 + Top3).

Time Complexity: $O(n)$. Since, we may need to adjust the third stack. Space Complexity: O(1).

Popping: For popping, we don't need to shift, just decrement the size of the appropriate stack.

Time Complexity: O(1). Space Complexity: O(1).

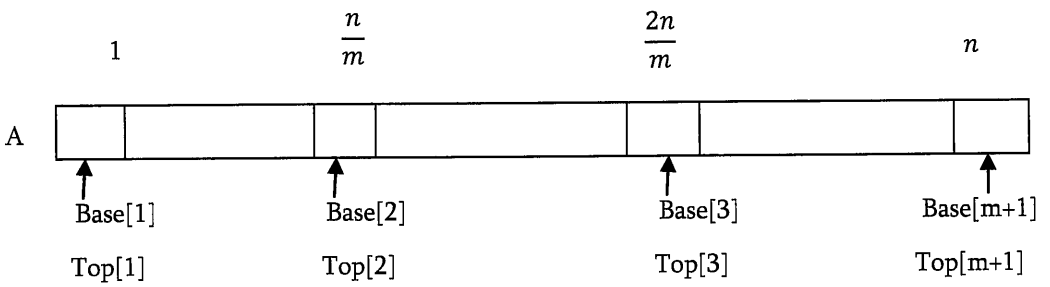
Problem-16 For Problem-15, is there any other way implementing middle stack?

Solution: Yes. When either the left stack (which grows to the right) or the right stack (which grows to the left) bumps into the middle stack, we need to shift the entire middle stack to make room. The same thing happens if a push on the middle stack causes it to bump into the right stack. To solve the above problem (number of shifts) what we can do is, alternating pushes could be added at alternating sides of the middle list (For example, even elements are pushed to the left, odd elements are pushed to the right).This would keep the middle stack balanced in the center of the array but it would still need to be shifted when it bumps into the left or right stack, whether by growing on its own or by the growth of a neighboring stack.

We can optimize the initial locations of the three stacks if they grow/shrink at different rates and if they have different average sizes. For example, suppose one stack doesn't change much. If you put it at the left then the middle stack will eventually get pushed against it and leave a gap between the middle and right stacks, which grow toward each other. If they collide, then it's likely you've run out of space in the array. There is no change in the time complexity but the average number of shifts will get reduced.

Problem-17 Multiple (m) stacks in one array: As similar to Problem-15, what if we want to implement m stacks in one array?

Solution: Let us assume that array indexes are from 1 to n. As similar to the discussion of Problem-15, to implement m stacks in one array, we divide the array into m parts (as shown below). The size of each part is $\frac{n}{m}$.



From the above representation we can see that, first stack is starting at index 1 (starting index is stored in Base[1]), second stack is starting at index $\frac{n}{m}$ (starting index is stored in Base[2]), third stack is starting at index $\frac{2n}{m}$ (starting index is stored in Base[3]) and so on. Similar to *Base* array, let us assume that *Top* array stores the top indexes for each of the stack. Consider the following terminology for the discussion.

- Top[i], for $1 \leq i \leq m$ will point to the topmost element of the stack i.
- If Base[i] == Top[i], then we can say the stack i is empty.
- If Top[i] == Base[i+1], then we can say the stack i is full.
  Initially Base[i] = Top[i] = $\frac{n}{m}(i-1)$, for $1 \leq i \leq m$.
- The i^{th}stack grows from Base[i]+1 to Base[i+1].

Pushing on to i^{th} stack:

1) For pushing on to the i^{th} stack, we check whether top of i^{th} stack is pointing to Base[i+1] (this case defines that i^{th} stack is full). That means, we need to see if adding a new element causes it to bump into the $i+1^{th}$ stack. If so, try to shift the stacks from $i+1^{th}$ stack to m^{th} stack towards right. Insert the new element at (Base[i] + Top[i]).
2) If right shifting is not possible then try shifting the stacks from 1 to $i-1^{th}$ stack towards left.
3) If both of them are not possible then we can say that all stacks are full.

```
void Push(int StackID, int data) {
        if(Top[i] == Base[i+1])
                Print i^th Stack is full and does the necessary action (shifting);
        Top[i] = Top[i]+1;
        A[Top[i]] = data;
}
```

Time Complexity: O(n). Since, we may need to adjust the stacks. Space Complexity: O(1).

Popping from i^{th} stack: For popping, we don't need to shift, just decrement the size of the appropriate stack. The only case to check is stack empty case.

```
int Pop(int StackID) {
        if(Top[i] == Base[i])
                Print i^th Stack is empty;
        return  A[Top[i]--];
}
```

Time Complexity: O(1). Space Complexity: O(1).

Problem-18 Consider an empty stack of integers. Let the numbers 1, 2, 3, 4, 5, 6 be pushed on to this stack only in the order they appeared from left to right. Let S indicates a push and X indicates a pop operation. Can they be permuted in to the order 325641(output) and order 154623? (If a permutation is possible give the order string of operations.

Solution: SSSXXSSXSXXX outputs 325641. 154623 cannot be output as 2 is pushed much before 3 so can appear only after 3 is output.

Problem-19 Earlier of this chapter, we have seen that, for dynamic array implementation of stack, we have used repeated doubling approach. For the same problem what is the complexity if we create a new array whose size is $n+K$ instead of doubling?

Solution: Let us assume that the initial stack size is 0. For simplicity let us assume that $K = 10$. For inserting the element we create a new array whose size is $0 + 10 = 10$. Similarly, after 10 elements we again create a new array whose size is $10 + 10 = 20$ and this process continues at values: 30, 40 ... That means, for a given n value, we are creating the new arrays at: $\frac{n}{10}, \frac{n}{20}, \frac{n}{30}, \frac{n}{40}$ ... The total number of copy operations are:

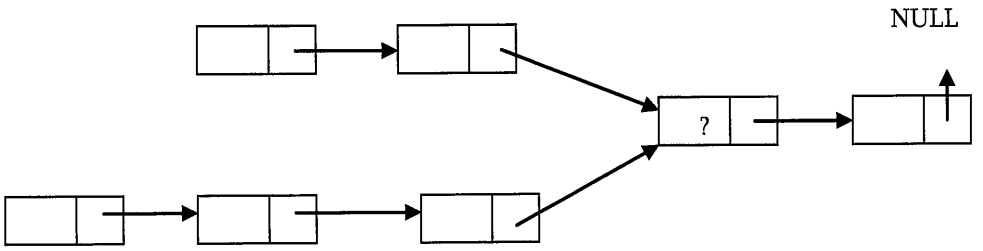
$$= \frac{n}{10} + \frac{n}{20} + \frac{n}{30} + \cdots 1 = \frac{n}{10}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right) = \frac{n}{10} logn \approx O(nlogn)$$

If we are performing n push operations, the cost of per operation is O($logn$).

Problem-20 Given a string containing n $S's$ and n $X's$ where S indicates a push operation and X indicates a pop operation, and with the stack initially empty, Formulate a rule to check whether a given string S of operations is admissible or not?

Solution: Given a string of length $2n$, we wish to check whether the given string of operations is permissible or not with respect to its functioning on a stack. The only restricted operation is pop whose prior requirement is that the stack should not be empty. So while traversing the string from left to right, prior to any pop the stack shouldn't be empty which means the no of $S's$ is always greater than or equal to that of $X's$. Hence the condition is at any stage on processing of the string, number of push operations (S) should be greater than number of pop operations (X).

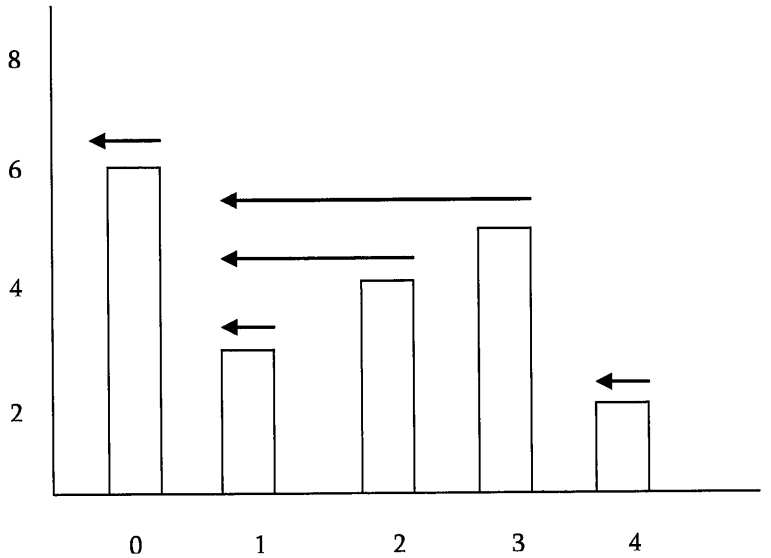
Problem-21 Suppose there are two singly linked lists both of which intersect at some point and become a single linked list. The head or start pointers of both the lists are known, but the intersecting node is not known. Also, the number of nodes in each of the list before they intersect are unknown and both list may have it different. $List1$ may have n nodes before it reaches intersection point and $List2$ might have m nodes before it reaches intersection point where m and n may be $m = n, m < n$ or $m > n$. Can we find the merging point using stacks?



Solution: **Yes.** For algorithm refer *Linked Lists* chapter.

Problem-22 **Finding Spans:** Given an array A the span $S[i]$ of $A[i]$ is the maximum number of consecutive elements $A[j]$ immediately preceding $A[i]$ and such that $A[j] \leq A[i]$?

Solution:



Day: Index i	Input Array A[i]	S[i]: Span of A[i]
0	6	1
1	3	1
2	4	2
3	5	3
4	2	1

This is a very common problem in stock markets to find the peaks. Spans have applications to financial analysis (E.g., stock at 52-week high). The span of a stocks price on a certain day, i, is the maximum number of consecutive days (up to the current day) the price of the stock has been less than or equal to its price on i. As an example, let us consider the following table and the corresponding spans diagram. In the figure the arrows indicates the length of the spans. Now, let us concentrate on the algorithm for finding the spans. One simple way is, each day, check how many contiguous days are with less stock price than current price.

Algorithm: FindingSpans(int A[],int n) {

```
        //Input: array A of n integers, Output: array S of spans of A
        int i, j, S[n]; //new array of n integers;
        for (i = 0; i < n; i++) {                          //Executes n times
                j = 1;                                     n
                while j <= i && A[i] > A[i-j]              1 + 2 + ...+ (n – 1)
                        j = j + 1;                         1 + 2 + ...+ (n – 1)
                S[i] = j;                                  n
        }
        return S;                                          1
}
```

Time Complexity: $O(n^2)$. Space Complexity: $O(1)$.

Problem-23 Can we improve the complexity of Problem-22?

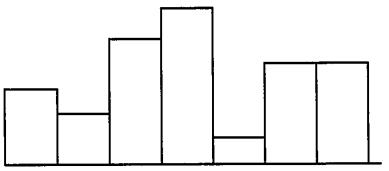
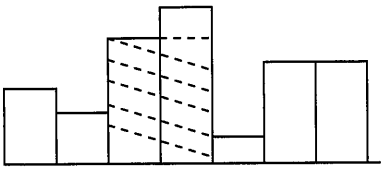
Solution: From the above example, we can see that the span $S[i]$ on day i can be easily calculated if we know the closest day preceding i, such that the price is greater than on that day than the price on day i. Let us call such a day as P. If such a day exists then the span is now defined as $S[i] = i - P$.

Algorithm: FindingSpans(int A[], int n) {

```
        struct stack *D = CreateStack();
        int P;
        for (int i = 0 i< n; i++) {
                while (!IsEmptyStack(D)) {
                        if(A[i] > A[Top(D)])
                                Pop(D);
                }
                if(IsEmptyStack(D))
                        P = -1;
                else    P = Top(D);
                S[i] = i-P;
                Push(D, i);
        }
        return S;
}
```

Time Complexity: Each index of the array is pushed into the stack exactly one and also popped from the stack at most once. The statements in the while loop are executed at most n times. Even though the algorithm has nested loops, the complexity is $O(n)$ as the inner loop is executing only n times during the course of algorithm (trace out an example and see how many times the inner loop is becoming success). Space Complexity: $O(n)$ [for stack].

Problem-24 **Largest rectangle under histogram:** A histogram is a polygon composed of a sequence of rectangles aligned at a common base line. For simplicity, assume that the rectangles are having equal widths but may have different heights. For example, the figure on the left shows the histogram that consists of rectangles with the heights 3, 2 , 5, 6, 1, 4, 4, measured in units where 1 is the width of the rectangles. Here our problem is: given an array with heights of rectangles (assuming width is 1), we need to find the largest rectabgle possible. For the given example the largest rectangle is the shared part.

Solution: A straightforward answer is to go for each bar in the histogram and find the maximum possible area in histogram for it. Finally, find the maximum of these values. This will require $O(n^2)$.

Problem-25 For Problem-24, can we improve the time complexity?

Solution: Linear search using a stack of incomplete subproblems: There are many ways of solving this problem. *Judge* has given a nice algorithm for this problem which is based on stack. Process the elements in left-to-right order and maintain a stack of information about started but yet unfinished sub histograms.

If the stack is empty, open a new subproblem by pushing the element onto the stack. Otherwise compare it to the element on top of the stack. If the new one is greater we again push it. If the new one is equal we skip it. In all these cases, we continue with the next new element. If the new one is less, we finish the topmost subproblem by updating the maximum area with respect to the element at the top of the stack. Then, we discard the element at the top, and repeat the procedure keeping the current new element. This way, all subproblems are finished until the stack becomes empty, or its top element is less than or equal to the new element, leading to the actions described above. If all elements have been processed, and the stack is not yet empty, we finish the remaining subproblems by updating the maximum area with respect to to the elements at the top.

```
struct StackItem {
        int height;
        int index;
};
int MaxRectangleArea(int A[], int n) {
        int i, maxArea=-1, top = -1, left, currentArea;
        struct StackItem *S = (struct StackItem *) malloc(sizeof(struct StackItem) * n);
        for(i=0; i<=n; i++) {
                while(top >= 0 && (i==n || S[top]→data > A[i])) {
                        if(top > 0)
                                left = S[top-1]→index;
                        else    left = -1;
                        currentArea = (i – left-1) * S[top]→data;
                        --top;
                        if(currentArea > maxArea)
                                maxArea = currentArea;
                }
                if(i<n) { ++top;
                        S[top]→data = A[i];
                        S[top]→index = i;
                }
        }
        return maxArea;
}
```

In first impression, this solution seems to be having $O(n^2)$ complexity. But if we look carefully, every element is pushed and popped at most once and in every step of the function at least one element is pushed or popped. Since the amount of work for the decisions and the update is constant, the complexity of the algorithm is $O(n)$ by amortized analysis. Space Complexity: $O(n)$ [for stack].